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Transmission and damping of hydromagnetic waves behind a strong shock front: implications for cosmic ray acceleration

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Summary. The transmission of hydromagnetic wave modes by a strong shock front is analysed. The character and damping rates of the low frequency modes in the high beta plasma behind the shock are described. These results are applied to the theory of Fermi acceleration by shock waves. It is shown that for a sufficiently large angle between the shock normal and the incident magnetic field, wave modes in the post-shock region are subject to linear transit time damping thus impairing the particle acceleration. Short-wavelength modes propagating along the field are subject to non-linear transit time damping (taking account of the saturation associated with particle trapping). Long-wavelength modes, responsible for scattering extremely relativistic ions, should not be damped and should thus mediate particle acceleration. If low-energy cosmic rays are to be accelerated by the Fermi process at a shock front, then either the wave turbulence ahead of the shock must have large amplitude (possibly invalidating the perturbation expansion upon which the calculations described in this paper are based) or hydromagnetic turbulence must be generated behind the shock front. The role of firehose instabilities in generating such post-shock turbulence is briefly discussed.

1 Introduction

Cosmic rays are observed to be accelerated by the first-order Fermi mechanism at interplanetary shock fronts (e.g. Kennel, Edmiston & Hada 1984). They are believed to be accelerated efficiently at interstellar and intergalactic shock waves, notably those associated with supernova remnants and double radio sources (see e.g. reviews by Axford 1981; Drury 1983; Blandford & Eichler 1985). In this mechanism, the particles are confined near the shock by frequent scattering off low-frequency Alfvén waves. This interaction takes place at the cyclotron resonance $\omega - k_{\|}v_{\|} \pm \Omega_i/\gamma = 0$, where ω is the wave frequency, $k_{\|}$ and $v_{\|}$ are the wave-number and particle velocity components along the ambient magnetic field, $\Omega_i = eB_0/m_i c$ is the ion cyclotron frequency and γ is the particle Lorentz factor. If the shock speed is much larger than the Alfvén speed the waves are essentially convected at the velocity of the background fluid. The waves on opposite sides of the shock front then approach one another and particles gain energy by being scattered backwards and forwards across the shock front.

As first described in detail by Bell (1978), the scattering waves in the *upstream region* can be generated self-consistently by the accelerated particles which try to stream down their density gradient, away from the shock. The anisotropy associated with this streaming motion leads to wave generation through a (linear) instability at the cyclotron resonance.

In the *downstream region*, the situation is less clear. In the simple theory there are no wave density gradients behind the shock and consequently no linear wave growth there. However, the waves generated upstream are convected into the shock since, by assumption, the Alfvén Mach number $M_A = V_1/a_1 \gg 1$. [Here, and in what follows indices 1 (2) will be used to indicate quantities in the upstream (downsteam) region of the shock; **V** is the fluid velocity, $a \equiv B/\sqrt{4\pi\varrho}$ the Alfvén speed, where B is the magnetic field strength, and ϱ the mass density of the fluid, and $s \equiv (\gamma^* P/\varrho)^{1/2}$, where γ^* is the specific heat ratio and P is the pressure.] Upon crossing the shock front, which is idealized as a plane discontinuity, the incident Alfvén waves generate 'reflected' and 'transmitted' waves *downstream*. It has generally been assumed (at least implicitly) that this is the origin of the downstream scattering centres.

In the simple test particle theory, the transmitted momentum space distribution function for the cosmic rays, $f_+(p)$ is related to the incident distribution function $f_-(p)$ is related to the incident distribution function $f_-(p)$ through

$$f_+(p) = qp^{-q} \int_0^p dp' p'^{(q-1)} f_-(p') \tag{1.1}$$

where the slope $q = 3r/(r-1)$ and r is the compression ratio of the shock front. Now, the source spectrum of galactic cosmic rays is believed to be a power law with $q \sim 4.2$ which implies that $r \approx 4.5$. This in turn implies that the sonic ($M = V_1/s_1$) and Alfvén Mach numbers are large and that the downstream plasma must be pressure-dominated, i.e. $\beta_2 = 8\pi P_{i2}/B_2^2 \gg 1$. (We assume that there is equipartition between ions and electrons behind the shock.) The propagation of Alfvén waves in high β plasmas differs from that in a cold plasma (Stepanov 1958; Foote & Kulsrud 1979). In particular, these modes are subject to quite strong damping by the thermal particles and it is important to determine whether or not shock acceleration will be inhibited through a shortage of post-shock scattering. This is the purpose of the present investigation.

In Section 2, we analyse the transmission of hydromagnetic disturbances by a strong shock front. In Section 3 we derive the dispersion relation for hydromagnetic waves propagating behind the shock and elucidate the conditions under which their damping is strong. In Section 4, we outline the implications of these results for the theory of cosmic ray acceleration at strong shock fronts. Improved derivations of the high β dispersion relation and the linear and non-linear transit time damping coefficients are contained in the three Appendices.

2 Transmission of Alfvén waves by a strong shock

We consider a shock in the y–z plane (Fig. 1) with the unperturbed flow along the magnetic field which makes an angle ϕ_1 (ϕ_2) with the shock normal in the upstream (downstream) region. It is always possible to transform to a frame with this flow configuration, provided $B_z \neq 0$. We assume that the flow is non-relativistic in this frame. The seven boundary conditions at the shock, continuity of mass, momentum and energy fluxes plus normal magnetic and tangential electric

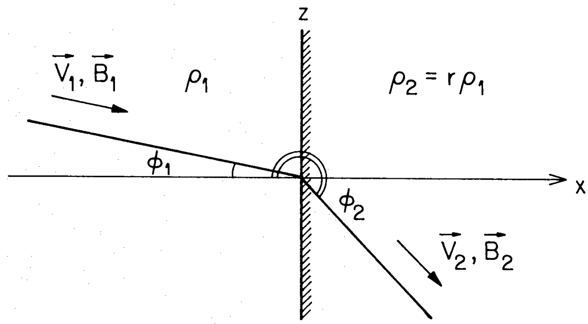

Figure 1. Physical quantities introduced in the text. The shock front lies in the $y-2$ plane. Quantities ahead of (behind) the shock have the subscript 1(2). We work in the frame where the fluid velocity **V** is parallel to the magnetic field **B**. The compression ratio r is the factor by which the density ϱ increases on crossing the shock front.

fields can be manipulated into the following set of equations (*cf.* Landau & Lifshitz 1960, section 54):

$$\varrho_1 V_{1n} = \varrho_2 V_{2n} = j \tag{2.1a}$$

$$B_{1n} = B_{2n} \equiv B_n \tag{2.1b}$$

$$j^2\left(\frac{1}{\varrho_1} - \frac{1}{\varrho_2}\right) = p_2 - p_1 + \frac{B_{2t}^2 - B_{1t}^2}{8\pi} \tag{2.1c}$$

$$V_{2t} - V_{1t} = \frac{B_n}{4\pi j}(B_{2t} - B_{1t}) \tag{2.1d}$$

$$\varepsilon_2 - \varepsilon_1 + ½\left(\frac{1}{\varrho_2} - \frac{1}{\varrho_1}\right)\left(p_2 + p_1 + \frac{(B_{2t} - B_{1t})^2}{8\pi}\right) = 0 \tag{2.1e}$$

$$B_n(V_{2t} - V_{1t}) - j\left(\frac{B_{2t}}{\varrho_2} - \frac{B_{1t}}{\varrho_1}\right) = 0 \tag{2.1f}$$

where $\varepsilon = p/(\gamma^* - 1)\varrho$ is the internal energy per unit mass and γ^* is the specific heat ratio. The subscripts n, t refer to components resolved perpendicular to and lying in the shock front respectively. This set of equations is valid regardless of the flow configuration and the value of the Mach numbers.

Under our assumptions of high Alfvén and sound Mach, numbers, the shock is almost purely hydrodynamical, and one finds:

$$\frac{\varrho_2}{\varrho_1} = \frac{V_{1n}}{V_{2n}} = \frac{\gamma^* + 1}{\gamma^* - 1} \equiv r \tag{2.2a}$$

$$B_{2t} = rB_{1t}; \qquad V_{2t} = V_{1t} \tag{2.2b}$$

$$p_2 \simeq \frac{r-1}{r} jV_{1n} \tag{2.2c}$$

to order $1/M_A^2$, $1/M^2$.

The perturbations induced in the downstream medium by the incident hydromagnetic waves can be obtained by perturbing the boundary conditions 2.1a–2.1f and linearizing in the

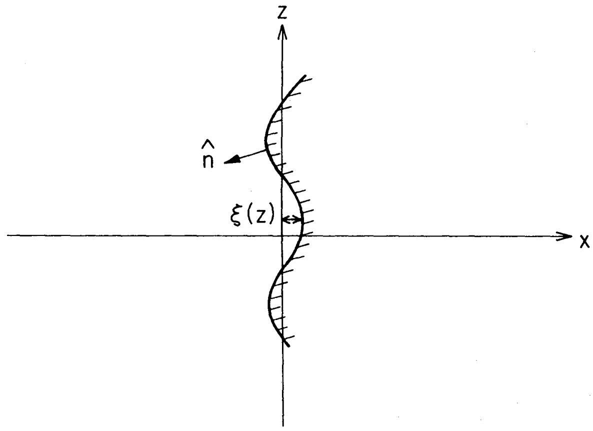

Figure 2. Motion of the shock front under the influence of an incident wave. The shock normal $\hat{\mathbf{n}}$ is locally rotated through an angle $-\partial\xi/\partial z$.

perturbations, e.g.

$$[\delta j]=[\delta\varrho V_{\rm n}+\varrho\delta V_{\rm n}]=0. \tag{2.3}$$

Some care must be taken, however, in defining the various perturbations to take into account the fact that the position of the shock is also perturbed, and the shock surface indulates, thereby rotating the shock normal locally (e.g. Anderson 1963). This is illustrated in Fig. 2. Assume that the displacement of the shock away from the $y-z$ plane depends only on z and is denoted by $\xi(z)$. In the linear approximation the shock normal is rotated through an angle $\partial\xi/\partial z$. Let us measure the field and velocity perturbation components along fixed coordinate axes (x, y, z):

$$\delta V_{\rm n}=\delta V_x-\left(\frac{\partial}{\partial t}+V_z\frac{\partial}{\partial z}\right)\xi$$

$$\delta\mathbf{V}_{\rm t}=\delta V_y\hat{\mathbf{y}}+\left(\delta V_z+V_x\frac{\partial\xi}{\partial z}\right)\hat{\mathbf{z}}$$

$$\delta B_{\rm n}=\delta B_x-B_z\frac{\partial\xi}{\partial z}$$

$$\delta B_{\rm t}=\delta B_y\hat{\mathbf{y}}+\left(\delta B_z+B_z\frac{\partial\xi}{\partial z}\right)\hat{\mathbf{z}}. \tag{2.4}$$

Also, the kinematical conditions

$$\omega_1=\omega_2\equiv\omega, \qquad \mathbf{k}_{2\rm t}=\mathbf{k}_{\rm t}=k_z\hat{\mathbf{z}} \tag{2.5}$$

must be satisfied by the waves, which are considered plane waves propagating in the frame of the shock with frequency ω and wave vector $\mathbf{k}$.

We now assume that Alfvén waves are incident on the shock from the upstream region. We limit the discussion to the case most relevant to shock acceleration, where the incident waves are circularly polarized, propagating along the magnetic field with their wave vector pointing away from the shock. This corresponds to the case where the waves are generated by the particles trying to stream away from the shock. We also confine our attention to the case $\omega\ll\Omega_{\rm i}/\beta$ for which the propagating modes behind the shock are similar to the modes in a low-β plasma (*cf.* Section 3 below).

When these waves strike the shock, each in principle generates six different waves in the downstream region: a fast magnetosonic wave propagating away from the shock, a reflected and a transmitted slow magnetosonic wave, propagating in opposite directions in the rest frame of the downstream fluid, similarly a reflected and transmitted Alfvén wave, and an entropy wave. Here, and in what follows, we assume $M_{A_1} \gg 1$, and consequently $\beta_2 \gg 1$. The properties of the different waves are summarized in Table 1. We have defined $\varpi \equiv \omega - \mathbf{k} \cdot \mathbf{V}$, $\hat{\mathbf{k}} \equiv \mathbf{k}/|\mathbf{k}|$, $\hat{\mathbf{b}} \equiv \mathbf{B}/|\mathbf{B}|$, $\hat{\mathbf{e}}_2 = \hat{\mathbf{b}} \times \hat{\mathbf{k}}/|\hat{\mathbf{b}} \times \hat{\mathbf{k}}|$. Furthermore, we have denoted the entropy per unit mass by σ. The relations between the perturbations in magnetic field, velocity, density, pressure, etc. and the dispersion relations themselves are a good approximation up to terms of order $1/M_A^2$, $1/M^2$.

Under these conditions, the perturbed boundary conditions, after some algebra, lead to the following relations between the perturbations associated with the waves in the up- and downstream region (neglecting small terms of order $1/M_A^2$, $1/M_S^2$, defining $r \equiv V_{1n}/V_{2n}$ and using the fact that the incident Alfvén waves have no density, pressure or entropy perturbation associated with them):

$$\delta \mathbf{V}_{2t} = \delta \mathbf{V}_{1t} + \frac{r-1}{4\pi j}(B_n \delta \mathbf{B}_{1t} + \mathbf{B}_{1t} \delta B_n) \tag{2.6a}$$

$$\delta \mathbf{B}_{2t} = r \delta \mathbf{B}_t; \qquad \delta B_{1n} = \delta B_{2n} \equiv \delta B_n \tag{2.6b}$$

$$\frac{\delta \varrho_2}{\varrho_2} = -(r-1)^2 \frac{\mathbf{B}_{1t} \cdot \delta \mathbf{B}_{1t}}{4\pi j V_{1n}} \tag{2.6c}$$

$$\frac{\delta p_2}{p_2} = \frac{2\delta V_n}{V_{1n}} - (r^2 + 2r - 1)\frac{\mathbf{B}_{1t} \cdot \delta \mathbf{B}_{1t}}{4\pi j V_{1n}} \tag{2.6d}$$

$$\delta \tilde{\sigma}_2 = \frac{2\delta V_{1n}}{V_{1n}} - \frac{r \mathbf{B}_{1t} \cdot \delta \mathbf{B}_{1t}}{2\pi j V_{1n}}. \tag{2.6e}$$

Equations (2.6a)–(2.6c) follow straightforwardly from (2.1d), (2.1b), (2.1f), and (2.2b), (2.1e) and (2.2a–c) respectively, (2.6d) follows from (2.1a) and (2.1c), whereas (2.6e) is derived from the definitions of the entropy per unit mass.

$$\sigma = C_v \ln\left(\frac{p}{\varrho^\gamma}\right) \equiv C_v \tilde{\sigma}. \tag{2.7}$$

where C_v is the specific heat at constant volume.

Table 1. Wave properties in a high β plasma for $k \ll \Omega_i/\alpha\beta$.

Alfvén mode	$\varpi = \pm \mathbf{k} \cdot \mathbf{a}$, $\delta \mathbf{B} = \delta B \hat{e}_2$ $\delta \mathbf{V} = \mp \frac{\delta \mathbf{B}}{\sqrt{4\pi\rho}}$, $\delta p = \delta \rho = \delta \sigma = 0$
Slow magnetosonic mode	$\varpi = \pm \mathbf{k} \cdot \mathbf{a}$, $\delta \mathbf{B} = \delta B \hat{e}_2 \times \hat{k}$ $\delta \mathbf{V} = \mp \frac{\delta \mathbf{B}}{\sqrt{4\pi\rho}}$, $\delta p = \delta \rho = \delta \sigma = 0$
Fast magnetosonic mode	$\varpi = \pm ks$, $\delta \mathbf{V} = \delta V \hat{k}$ $\delta \rho = \pm \frac{\delta V}{s}\rho$, $\delta p = \pm \gamma^* p \frac{\delta V}{s}$ $\delta \mathbf{B} = \frac{\delta \rho}{\rho} B(\hat{k} \cdot \hat{b})\hat{e}_2 \times \hat{k}$ $\delta \sigma = 0$
Entropy mode	$\varpi = 0$, $\delta \mathbf{B} = \delta \mathbf{V} = \delta p = 0$ $\delta \rho = -(\delta \sigma / \gamma^* c_V)\rho$

Equations (2.4), (2.6) are sufficient to determine the seven quantities that determine the amplitude of the downstream waves and the displacement of the shock; they are $\delta B_{\rm A}^T$, $\delta B_{\rm A}^R$, $\delta B_{\rm S}^T$, $\delta B_{\rm S}^R$, δp_2, $\delta\sigma_2$, and ξ, the amplitude of the 'transmitted' and 'reflected' magnetic field perturbations associated with the Alfvén and slow modes respectively, the pressure perturbation of the fast mode propagating away from the shock (which is the only one allowed), the entropy perturbation associated with the entropy wave and the displacement of the shock. In this context 'reflected' and 'transmitted' are to be interpreted as meaning waves which in the rest-frame of the downstream fluid propagate in the opposite or same direction as the upstream Alfvén waves in the upstream fluid rest frame. These quantities, together with wave properties listed in Table 1 and the kinematical conditions (5) for the wave frequency and component of the wave vector tangential to the shock, are sufficient to specify the downstream state.

2.1 PARALLEL SHOCK

We first consider the relatively simple case where the magnetic field is perpendicular to the shock front, i.e. $B_{1\rm t}=B_{2\rm t}=0$. This automatically implies that $\delta B_{1\rm n}=0=\delta V_{1\rm n}$ because of the polarization characteristics of the incoming Alfvén waves. The set of equations (2.6) which relate the up- and downstream perturbations then simply reduce to

$$\delta\mathbf{V}_{2\rm t}=\delta\mathbf{V}_{1\rm t}+\frac{r-1}{M_{\rm A_1}}\frac{\delta\mathbf{B}_{1\rm t}}{\sqrt{4\pi\varrho_1}}+\mathrm{O}\left(\frac{1}{M_{\rm A}^2}\right)$$

$$\delta\mathbf{B}_{2\rm t}=r\delta\mathbf{B}_{1\rm t}+\mathrm{O}\left(\frac{1}{M_{\rm A}^2}\right) \tag{2.8}$$

with all other downstream quantities, and the shock displacement, ξ, vanishing. If the wave vector is indeed along the magnetic field for the incident waves, this will also be true for the downstream waves. This is the degenerate case where one cannot make a distinction between the Alfvén and the slow mode and there are only two waves downstream: the transmitted and reflected Alfvén wave.

In the case of a *monochromatic wave*, one has upstream:

$$\delta\mathbf{B}_1=\mathbf{b}_k\exp(i\mathbf{k}\cdot\mathbf{x}-i\omega t)+\text{c.c.}$$

$$\omega_1=\mathbf{k}\cdot(\mathbf{V}-\mathbf{a})_1, \qquad \delta\mathbf{V}_1=-\delta\mathbf{B}_1/\sqrt{4\pi\varrho_1}. \tag{2.9}$$

Here 'c.c.' denotes complex conjugation. In the downstream region one has:

$$\delta\mathbf{B}_2=\sum_{\sigma=T,R}\mathbf{b}_k^\sigma\exp(i\mathbf{k}^\sigma\cdot\mathbf{x}-i\omega^\sigma t)+\text{c.c.}$$

$$\omega^T=\mathbf{k}^T\cdot(\mathbf{V}-\mathbf{a})_2, \qquad \omega^R=\mathbf{k}^R\cdot(\mathbf{V}+\mathbf{a})_2$$

$$\delta\mathbf{V}_k^T=-\frac{\mathbf{b}_k^T}{\sqrt{4\pi\varrho_2}}, \qquad \delta\mathbf{V}_k^R=+\frac{\mathbf{b}_k^R}{\sqrt{4\pi\varrho_2}}. \tag{2.10}$$

Equations (2.4) and (2.5) can then be solved to yield:

$$b_k^{T,R}=\tfrac{1}{2}\left\{r\pm\sqrt{r}\left(1-\frac{r-1}{M_{\rm A_1}}\right)\right\}b_k$$

$$k^{T,R}=rk\left(\frac{M_{\rm A_1}-1}{M_{\rm A_1}\mp\sqrt{r}}\right). \tag{2.11}$$

Here the upper (lower) sign on the right-hand side refers to the transmitted (reflected) wave. We have used the fact that the unperturbed flow is field-aligned. The equivalent of these equations was first derived by McKenzie & Westphal (1969).

In the case of waves generated in the process of shock acceleration, one has a *continuous spectrum* of waves; and (2.8) is modified to $\delta\mathbf{B}_1=\int d^3k(2\pi)^{-3}\mathbf{b}_k\exp(i\mathbf{k}\cdot\mathbf{x}-i\omega_k t)$. In the transmission and reflection coefficients one has to take into account the change in wavenumber across the shock, introducing an extra factor $\partial(\mathbf{k})/\partial(\mathbf{k}^{T,R})$, the Jacobian of the transformation between the up- and downstream wave numbers. In that case one finds $b_k^T\equiv T_{\mathrm{A}}b_k$, $b_k^R\equiv R_{\mathrm{A}}b_k$, now evaluated at the *same* wavenumber, where:

$$T_{\mathrm{A}}=\frac{1}{2r}(r+\sqrt{r})-\frac{r-1}{\sqrt{r}M_{\mathrm{A}_1}}+\mathrm{O}\left(\frac{1}{M_{\mathrm{A}_1}^2}\right)$$

$$R_{\mathrm{A}}=\frac{1}{2r}(r-\sqrt{r})+\frac{r-1}{\sqrt{r}M_{\mathrm{A}_1}}+\mathrm{O}\left(\frac{1}{M_{\mathrm{A}_1}^2}\right). \tag{2.12}$$

When $M_{\mathrm{A}_1}\gg 1$, and a strong shock in a monoatomic gas $[r=(\gamma^*+1)/(\gamma^*-1)=4]$, one has $T\simeq 3/4$, $R=1/4$. The energy density of the downstream waves, which for Alfvén waves equals $W=\int\{d^3k/(2\pi)^3\}(|b_k|^2/4\pi)$, increases by a factor

$$\alpha=\frac{W^T+W^R}{W_1}=T_{\mathrm{A}}^2\frac{\partial k^T}{\partial k}+R_{\mathrm{A}}^2\frac{\partial k^R}{\partial k}\simeq\frac{r+1}{2}+\mathrm{O}\left(\frac{1}{M_{\mathrm{A}_1}}\right) \tag{2.13}$$

across the shock ($\alpha=5/2$ for a strong shock). Note that the wave energy flux is changed by a factor of α/r($=5/8$ for a strong shock).

2.2 OBLIQUE SHOCK

The case of an oblique shock, where both $\mathbf{B}_{1\mathrm{t}}$ and δB_{n} are non-vanishing, is much more complicated. As seen from equations (2.6) and (2.7), there are now density and entropy perturbations downstream, indicating the presence of an entropy wave and fast magnetosonic (sound) wave downstream. Furthermore, in principle, the shock itself will be displaced ($\xi\neq 0$).

Assuming again that the incident Alfvén waves propagate away from the shock along the magnetic field and are circularly polarized, we expand these waves into two sets of plane

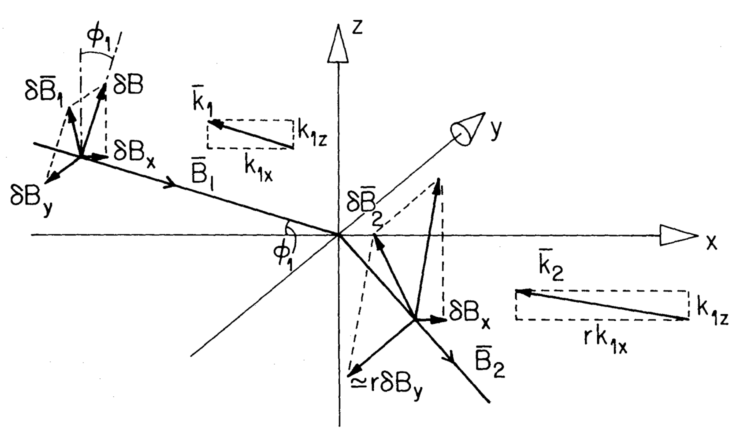

Figure 3. Magnetic field amplitudes at a strong shock. An Alfvén wave with **k** antiparallel to **B** is incident on the shock from $x<0$. Upon crossing the shock it is converted into a 'reflected' and a 'transmitted' Alfvén wave with magnetic amplitude along Oy and a pair of magnetosonic modes polarized in the x–z plane.

polarized waves, their phases differing by 90°. One set has its $\delta\mathbf{B}$ vector in the plane containing $\mathbf{B}_1$, $\mathbf{k}_1$ and the shock normal (the $x-z$ plane, see Fig. 3), the other has its magnetic field perturbation perpendicular to that, essentially parallel to the plane of the shock (the $y-z$ plane).

As can be seen from (2.4) and (2.6), the latter set of waves, with field and velocity perturbations δB_y, δV_y still have $\delta B_n=0$; and behave the same as waves incident on a normal shock, since their polarization is such that they do not 'see' that the shock is oblique. Consequently they are the source of 'reflected' and 'transmitted' Alfvén waves downstream, with amplitudes

$$b_{\mathrm{A}}^{T}(k)=T_{\mathrm{A}}b_y^{I}(k)$$

$$b_{\mathrm{A}}^{R}(k)=R_{\mathrm{A}}b_y^{I}(k) \tag{2.14}$$

where T_{A} and R_{A} are given in (2.12), and $\mathbf{b}^I$ is the amplitude of Fourier component of the spectrum of incoming waves.

The component of the waves with $\delta\mathbf{B}_1$ in the $x-z$ plane must be converted, upon crossing the shock, into 'reflected' and 'transmitted' slow magnetosonic modes a fast mode and an entropy wave. [An entropy wave is a non-adiabatic, isobaric perturbation convected with the fluid (Akhiezer *et al.* 1975, p. 87).] This is because the normal magnetic field perturbation, δB_n is continuous at the shock, and so the waves in the downstream medium generally have a non-vanishing value of $(\delta\mathbf{B}\cdot\mathbf{B})_2$. This only happens in the slow (and fast) magnetosonic mode. It is this component of the Maxwell stress tensor that is responsible for displacing the shock from its equilibrium position.

The actual expressions for the various reflection and transmission coefficients are cumbersome. To zero order in $1/M_{\mathrm{A}_1}$, some simplification results from the fact that $\delta\varrho_2/\varrho_2\simeq 0$ and the fact that the wave vector for the transmitted and reflected slow mode are the same $\mathbf{k}^{T,R}\sim[rk_{1x},\ 0,\ k_{1z}]$. Furthermore, the magnetic perturbation due to the fast mode can be neglected.

$$\delta B^{\mathrm{F}}=\mathrm{O}\left(\frac{\delta V^{\mathrm{F}}}{s_2}\right)B_2\simeq\left(\frac{a_2}{s_2}\right)\delta B_2=\mathrm{O}\left(\frac{1}{M_{\mathrm{A}_1}}\right)\delta B_2. \tag{2.15}$$

The entropy wave has no magnetic perturbation. Hence, the downstream slow mode has vanishing divergence, $\mathbf{k}_2^S\cdot\delta\mathbf{B}_2=0$, with k_2^S the wave vector of the downstream slow modes. In other words,

$$\frac{\delta B_{2x}}{\delta B_{2z}}=\tan\phi_1/r. \tag{2.16}$$

The set of boundary conditions (2.6b) can now be written as:

$$\delta B_{2z}-(r-1)\,B_{1z}\frac{\partial\xi}{\partial z}=r\delta B_{1z}=r\delta B_1\cos\phi_1$$

$$\delta B_{2x}-(r-1)\,B_{1z}\frac{\partial\xi}{\partial z}=\delta B_{1x}=\delta B\sin\phi_1. \tag{2.17}$$

Equations (2.16) and (2.17) can only be sastisfied simultaneously if $\partial\xi/\partial z\,(=ik_z\xi)=0$. So, again to lowest order in $1/M_{\mathrm{A}_1}$, the shock is not displaced. The other relevant boundary conditions then become

$$\delta V_{2z}=\delta V_{1z}=\frac{\delta B}{\sqrt{4\pi\varrho_1}}\cos\phi_1 \tag{2.18a}$$

$$\delta V_{2x} = \frac{\delta V_{1x}}{r} = \frac{\delta B_1}{\sqrt{4\pi\varrho_1}} \frac{\sin\phi_1}{r}, \tag{2.18b}$$

where the last equation follows from subtracting (2.6d) and (2.6e) neglecting terms of order $1/M_{A_1}$. This result immediately implies that to this order there cannot be a fast mode downsteam, since $\mathbf{k}_2^S \cdot \delta\mathbf{V}_2 = \mathbf{k}_1 \cdot \delta\mathbf{V}_1 = (\mathbf{k}_1 \cdot \delta\mathbf{B}_1)/\sqrt{4\pi\varrho_1} = 0$, while $(\mathbf{k}_2^S \cdot \mathbf{k}_2^F) \neq 0$. where $\mathbf{k}_2^F$ is the wave-vector of the downstream fast mode. Combining (2.16), (2.17), and (2.18) then yields the following expression for the reflection and transmission coefficients, again assuming a continuous spectrum of incident circularly polarized waves (again situated at the *same* wavenumber):

$$b_S^T(\mathbf{k}) = T_S b(k)/\sqrt{2}$$

$$b_S^R(\mathbf{k}) = R_S b(k)/\sqrt{2}$$

where

$$T_S = \frac{\sqrt{r^2\cos^2\phi_1 + \sin^2\phi_1}}{2r}\left(1 + \frac{1}{\sqrt{r}}\right)$$

$$R_S = \frac{\sqrt{r^2\cos^2\phi_1 + \sin^2\phi_1}}{2r}\left(1 - \frac{1}{\sqrt{r}}\right). \tag{2.19}$$

When $\phi_1 \to 0$, one recovers the result for a parallel shock.

A further important effect introduced when the shock is oblique is that the downsteam Alfvén and magnetosonic waves propagate at a finite angle with respect to the ambient magnetic field, i.e. $\mathbf{k} \times \mathbf{B} \neq 0$. Specifically one has in the downsteam medium:

$$\begin{aligned}(k_z B_x - k_x B_z)_2 &\cong \frac{|k_1|}{B_1}(r^2 - 1)\, B_{1x} B_{1z} \\ &\cong -|k_1|\, B_1 (r^2 - 1) \sin\phi_1 \cos\phi_1. \end{aligned} \tag{2.20}$$

As will be shown below, this is of importance to the wave damping in downsteam region.

To summarize this section, we have calculated the interaction between a strong shock and incident low-frequency Alfvén waves propagating backwards along the magnetic field. Upon crossing an oblique shock, the (circularly) polarized Alfvén waves are converted into plane polarized 'reflected' and 'transmitted' Alfvén waves and slow magnetosonic waves, which carry the downstream perturbation in the magnetic field in two (approximately orthogonal) directions. The (small) pressure and density perturbations are carried by a fast magnetosonic (sound) wave and an entropy wave. These last two waves are unimportant for the scattering of accelerated particles off magnetic irregularities, and will not concern us further. What is significant is that the waves behind an oblique shock wave propagate at a finite angle to the ambient field. Although we have not explicitly calculated the transmission properties of waves with $\Omega_i/\beta \lesssim \omega \lesssim \Omega_i$ these general remarks apply to this case as well.

3 Wave damping in a high β plasma

As was argued in Section 1, the downstream medium of a high Alfvén–Mach number, strong shock is in a high β state. The propagation of low-frequency Alfvén and magnetosonic waves in a high β plasma has been analysed by Foote & Kulsrud (1979) following earlier work by Stepanov (1958). They showed that the phase velocity of the propagating eigenmodes differed substantially from the Alfvén speed above a frequency Ω_i/β. Although modes propagating along the field are undamped, obliquely propagating waves are subject to substantial transit time damping.

The properties of these waves are discussed in detail in Appendix A. Here we use the notation $\varkappa \equiv kr_L = kv_i/\Omega_i$, $v_i = \sqrt{2T_i/m_i}$. For waves gyroresonant with fast particles, the resonance condition yields $\varkappa \simeq v_i/\gamma v$. We are interested in the propagation of waves resonant with suprathermal particles if the particle speed is v and $\gamma = (1 - v^2/c^2)^{-1/2}$, then the relevant dimensionless k-vector is $\varkappa = v_i/\gamma v \ll 1$. Waves with $\varkappa \gtrsim 1$ will be rapidly damped by gryoresonance with the thermal ions (e.g. Achterberg 1981a, b). We also characterize the mode by the angle θ its propagation direction makes with the magnetic field direction.

We are interested in the intensity damping rate $\gamma_L = -2\,\mathrm{Im}(\omega')$ where ω' refers to the wave frequency in the plasma rest frame. Henceforth we measure all such damping rates in units of the ion Larmor frequency, i.e. $\Gamma_L \equiv \gamma_L/\Omega_i$. In Table 2, we summarize the properties of the different wave modes given asymptotic fomulae for the real and imaginary parts of ω'. These expressions are asymptotically valid as $\beta \to \infty$ and (for given β) are the leading terms in expansions derived in specific regions of the $\varkappa, \theta$ plane. In order to keep the algebra tractable, we make the small angle approximation $\theta \lesssim 1$. The quoted expressions are invalid for $\theta \sim 90°$. However, the waves are strongly damped in this limit anyway.

Numerical solution of the full dispersion relation shows that they are accurate to a factor $\lesssim 2$ for all $\varkappa \lesssim 1$ and for $\beta \gtrsim 30$ except in the immediate vicinity of cut-offs. The propagating modes can be labelled as Alfvénic and magnetosonic modes. Four distinct non-propagating or reactive modes can also be found under the specified conditions. The most slowly damped mode (for given $\varkappa$, θ, β) is the Alfvén mode except when $1.8\theta^2 \gtrsim \varkappa \gtrsim 2.8\beta^{-1/2}$ in which case the first reactive mode persists longer. These results are summarized in Fig. 4. An approximate interpolating formula, good to factor 2 and valid near cut-offs, for the slowest damping rate in a high β plasma is

$$\Gamma_L \equiv \frac{\gamma_L}{\Omega_i} = \frac{\pi^{1/2}\varkappa\theta^2}{(1+\varkappa^2\beta/8)(1+4\pi\theta^4/\varkappa^2)}. \tag{3.1}$$

In Appendix B we demonstrate that the strongest linear damping mechanism is transit-time damping. The magnetosonic branch is more rapidly damped than the Alfvén branch because there is a larger component of the wave magnetic field resolved along the ambient field.

Waves that propagate nearly along the field are only subject to weak linear damping since $\delta\mathbf{B}\cdot\mathbf{B}_0 \sim 0$. However, they do induce changes in B of order $|\delta\mathbf{B}|^2$: $\delta B \simeq |\delta\mathbf{B}|^2/2B_0$ and are subject to non-linear damping. The wave–particle interaction now takes place at the Cherenkov resonance of the particle with the high- and low-frequency beat wave of two Alfvén waves or slow magnetosonic waves propagating along the field (all frequencies are taken to be positive):

$$\omega \pm \omega' - (k_\parallel \pm k_\parallel')\,V_\parallel = 0. \qquad 3.2$$

As in the case of linear transit time damping, this implies that $dp_\perp/dt = 0$. The corresponding damping rate of the waves, which is calculated in Appendix C, equals:

$$\Gamma_{NL} \simeq \left(\frac{\pi}{2}\right)^{1/2} \frac{|k_\parallel|\,v_i}{\Omega_i} \int_0^{k_\parallel} d\ln k I_m(k). \tag{3.3}$$

Here $I_m(k)\,d\ln k$ is the *magnetic* energy density of the waves between k and $k+dk$ divided by the energy density in the ambient magnetic field $B_0^2/8\pi$. This expression, up to factors of order unity, is correct for damping of Alfvén and slow magnetosonic waves in the long-wavelength regime, as well as the damping of fast waves and slow waves by fast waves in the short-wavelength regime. Obviously it does not apply to the non-propagating reactive modes. For Alfvén waves, non-linear transit-time damping is the dominant damping mechanism. We assume that the dominant

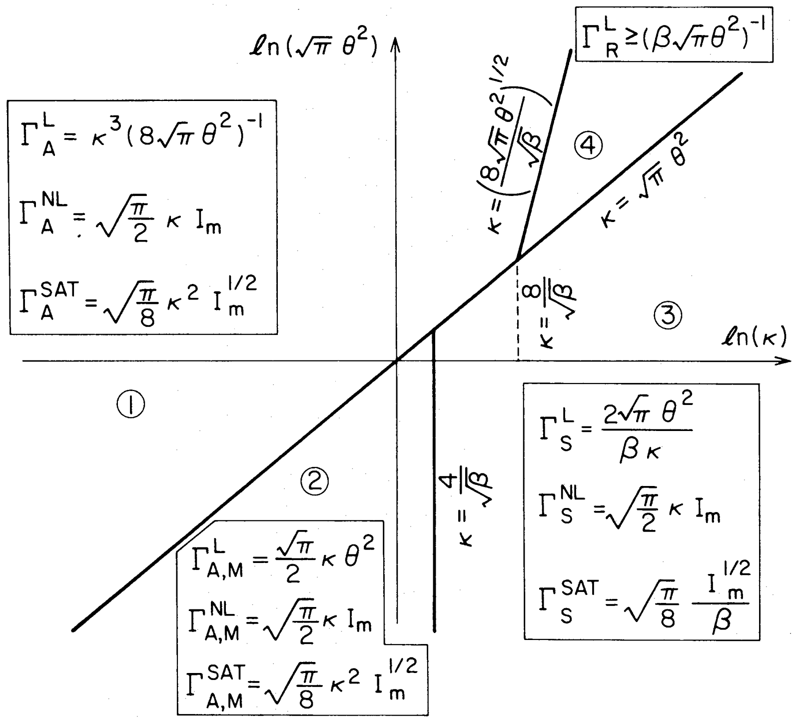

Figure 4. The linear (L), non-linear (NL), and saturated damping (SAT) rates in units of the ion gyrofrequency for the slowest linearly damped modes as a function of $\varkappa=(V_i/\gamma V)$ and $\sqrt{\pi}\theta^2$. Different regions corresponding to different limiting solutions of the dispersion relation (A3) are separated by bold lines. An isothermal plasma with $T_\perp = T_\parallel$ has been assumed. The subscripts A, M, S, and R designate Alfvén magnetosonic, slow, and reactive modes respectively.

contribution to the integral in equation (3.3) comes from waves with wavelength similar to that of the mode being damped. We then estimate

$$\Gamma_{NL} \simeq \left(\frac{\pi}{2}\right)^{1/2} \varkappa I_m. \tag{3.4}$$

Equation (3.3) agrees with a calculation of non-linear Landau damping (of which this is the high β limit) for the case of waves propagating along the magnetic field (Lee & Völk 1973; Achterberg 1981b; Völk & Cesarsky 1982), but our derivation is also correct for linearly polarized waves with $k_1 \neq 0$. However, as pointed out by Völk & Cesarsky (1982), the non-linear damping rate will saturate when thermal ions are trapped in the potential well created by the heat wave for a coherence time

$$t_{coh} \simeq 2\pi \left| k\left(\frac{\omega}{k} - \frac{\partial\omega}{\partial k}\right)\right|^{-1}.$$

The saturated damping rate roughly equals $\Gamma_{SAT} = \Gamma_{NL}(t_{bounce}/t_{coh})$, where $t_{bounce} \simeq (2\pi/\Omega_i)\varkappa^{-1} I_m^{-1/2}$ is the bounce period for the trapped particles. The long-wavelength Alfvén and magnetosonic mode, as well as the fast mode have $t_{coh} \simeq (2\pi/\Omega_i)\varkappa^{-2}$, whereas the slow mode has $t_{coh} \simeq (2\pi\beta/\Omega_i)$. This yields:

$$\Gamma_{sat} = \left(\frac{\pi}{8}\right)^{1/2} \varkappa^2 I_m^{1/2} \times \begin{cases} 1 & \text{A, MS, F} \\ 1/\beta\varkappa^2 & \text{Slow} \end{cases} \tag{3.5}$$

provided $I_m \gtrsim \varkappa^2$, $(1/\varkappa^2\beta^2)$ for the Alfvén, magnetosonic, fast (slow) modes, for the non-propagating 'reactive' modes their conclusions of course do not hold.

We note here that at low frequencies, the non-linear Landau damping affects the Alfvén waves and parallel propagating slow modes separately. Since the interaction of the incident Alfvén waves with the shock produces waves of the two modes in the downstream medium whose magnetic perturbations are almost orthogonal, no beat waves of 'mixed origin' result.

Damping of Alfvén waves due to three-wave interaction with the rapidly damped slow modes: $A+A'\rightarrow S$ is negligible by comparison. Applying the general result of Akhiezer *et al.* (1975, p. 110) to calculate the coupling strength in the high β case, we find a damping rate for the Alfvén waves due to three wave interactions:

$$\Gamma_{\mathrm{A+A'\rightarrow S}}\simeq\frac{\pi}{4}\frac{a^4}{s^4}\frac{|k_\parallel|a}{\Omega_\mathrm{i}}\sin^2\theta\frac{|k_\parallel|W_\mathrm{A}(k_\parallel)}{U_\mathrm{M}} \tag{3.7}$$

which is smaller than the damping rate due to non-linear transit-time damping, roughly by a factor $\propto\beta^{-5/2}\ll1$. Three-wave interactions between the Alfvén waves has zero interaction strength.

To summarize this section, we have presented the linear damping rates of the modes likely to exist behind a strong shock (equation 3.1). We have also calculated the non-linear damping rate for nearly parallel modes (equation 3.3) and corrected this for the effects of particle trapping where appropriate (equation 3.5).

4 Application to shock wave acceleration

We now turn to the question of whether or not post-shock damping of Alfvén and magnetosonic waves can inhibit first-order Fermi acceleration at a shock front. We confine our attention to strong (M, $M_\mathrm{A}\gg1$) collisionless shocks propagating through a fully ionized plasma with $\gamma=5/3$. It is believed that the majority of galactic cosmic rays are accelerated in this manner.

Under these conditions,

$$\beta_2=\frac{2(r-1)M_\mathrm{A}^2}{r[1+(r^2-1)\sin^2\phi_1]}. \tag{4.1}$$

Likewise the angle between the post-shock field and the shock normal satisfies

$$\tan\phi_2=r\tan\phi_1. \tag{4.2}$$

The ion thermal speed behind the shock is

$$v_\mathrm{i}=\frac{(r-1)^{1/2}}{r}V_1\cos\phi_1. \tag{4.3}$$

For the shocks under consideration, we expect that $30\lesssim\beta_2\lesssim3000$.

Consider low-frequency Alfvén modes behind the shock, polarized in the $\hat{\mathbf{y}}$ direction. Using the transmission coefficients expressed by equations (2.10) and (2.11), it can be shown that the transmitted and reflected waves propagate along directions that make angles θ_T, θ_R with the ambient field where

$$\tan\theta_\mathrm{T}=\frac{(r^2-1)\sin\phi_1\cos\phi_1}{r} \tag{4.4}$$

$$\tan\theta_R=\frac{2r^2\tan\phi_1-(r^2-1)\sin\phi_1\cos\phi_1}{r}. \tag{4.5}$$

Unless the shock is almost parallel ($\phi_1 \simeq 0$) or very weak $r \sim 1$, the transmitted waves make a substantial angle ($\theta \sim 1$) with the downstream field. As an illustration, set $r=4$, $\phi_1=60°$, to obtain $\theta_T=58°$ and $\theta_R=85°$. We set $\theta \sim 1$ when estimating damping rates.

We next ask if the waves behind the shock can scatter the cosmic rays before they are damped. It is convenient to define a diffusion 'rate' $\Gamma_{dif}=V_2^2/D_2$, which is roughly the reciprocal of the time it takes the waves to be convected through those resonant cosmic rays that will return to the shock front and are still being accelerated. The spatial diffusion coefficient D_2 depends inversely upon the resonant wave intensity, which will in turn depend upon the transmission coefficient (Bell 1978); with a pitch angle scattering frequency $\nu_S \simeq (\pi/8)(\Omega_i/\gamma) I_m$ due to gyroresonant interactions one can estimate:

$$\Gamma_{dif} \simeq 3 \frac{V_2^2}{V^2} \frac{\nu_S}{\Omega_i} \simeq \frac{3\pi}{4} I_m \varkappa^2 \gamma. \tag{4.6}$$

Efficient particle acceleration require that the diffusion rate exceed the associated wave damping rates given in Table 2 and by equation (3.1). Under astrophysical conditions, we anticipate that $\beta_1 \sim 1$. The upstream waves can be regarded as Alfvén waves propagating along the field. The downstream modes will therefore propagate to an angle $\theta_2 \simeq \tan^{-1}(r \sin \phi_1)$ with respect to the downstream field direction.

In Fig. 4 we have collected all the relevant results in the different regimes. The linear, non-linear, and saturated damping rates Γ_L, Γ_{NL}, and Γ_{sat} have been calculated for the modes *least* damped linearly. This figure should be compared with Fig. 5, which describes the dispersion relation an anisotropic plasma. We have used the resonance condition to express the wavenumbers in particle variables $\varkappa=V_i/\gamma V$ (measured downstream), and equations (4.4–5) to express the propagation angle θ in terms of $r \sin \phi_1$, assuming $\theta \lesssim 1$.

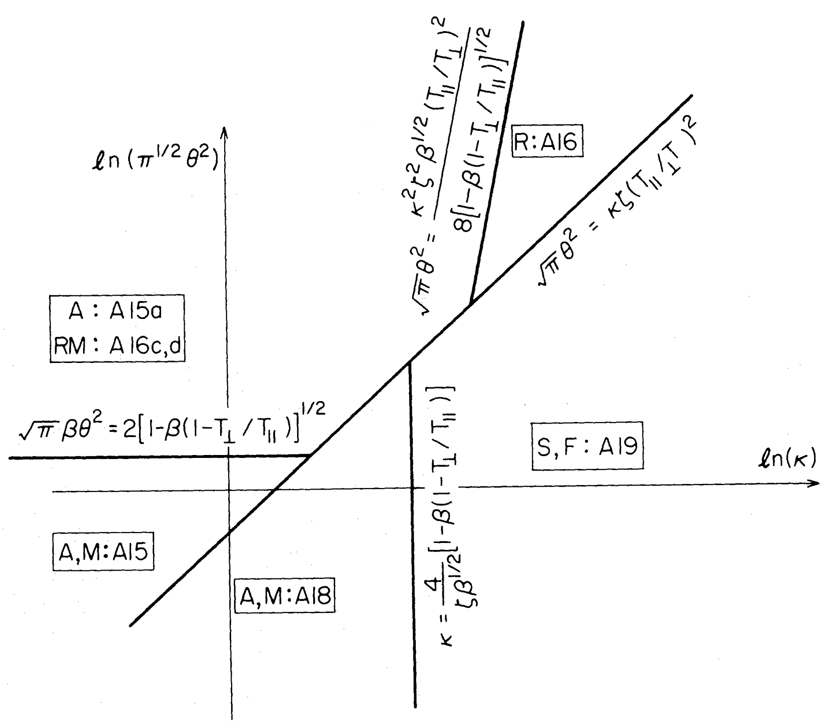

Figure 5. Modes in an anisotropic high β plasma. Alfvén (A), magnetosonic (M), reactive (R), slow (S), and fast (F) modes are identified together with relevant equation numbers describing their dispersion relations. Different regimes are separated by bold lines.

As a particle is accelerated, $\varkappa$ decreases. The diffusion rate $\Gamma_{\rm dif}=(3\pi/4)\gamma I_{\rm m}\varkappa^2$ decreases steadily as long as $I_{\rm m}$ decays slower than $\varkappa^{-2}(\varkappa^{-3})$ for non-relativistic (relativistic) particles. When the upstream wave turbulence is strong $[I_{\rm m}=O(1)]$ the non-linear or saturated damping rates apply in all regions except in region 4 (reactive modes). If the unsaturated non-linear damping rate applies, we obtain $\Gamma_{\rm dif}/\Gamma_{\rm NL}\simeq V_{\rm i}/V\ll 1$. This implies that when the scattering waves cross the shock front, they are damped within a downstream diffusion length of the accelerated particles, thus inhibiting the shock acceleration mechanism.

The saturated damping rate will apply when $t_{\rm bounce}\lesssim t_{\rm coh}$, i.e. when $I_{\rm m}\lesssim\varkappa^2$ (regime 1 and 2) or $I_{\rm m}>\varkappa^2\beta^{-2}$ (regime 3). One then has:

$$\frac{\Gamma_{\rm dif}}{\Gamma_{\rm sat}}=3\sqrt{\pi}/8\gamma I_{\rm m}^{1/2}\times\begin{cases}1 & \text{regions 1, 2}\\ \beta\varkappa^2 & \text{region 3.}\end{cases} \tag{4.7}$$

This implies that low-energy particles can be accelerated, starting in the short-wavelength region 3 when $\varkappa=O(1)$, to relativistic energies provided $I_{\rm m}(\varkappa)\gtrsim 0.3/\gamma^2$, requiring rather strong magnetic turbulence around $\gamma=1$.

For supernova shocks propagating in a hot interstellar medium with $\beta\sim 1$, we expect the *upstream* turbulence to saturate at a value $I_{\rm m_1}\simeq|(\delta B/B)|_1^2\sim 0.3$. Upon crossing the shock, the component of wave magnetic field polarized in the plane perpendicular to the shock normal and ambient magnetic field survives as downstream Alfvén waves, with dimensionless energy density $(I_{\rm m})_2\simeq(r+1)(I_{\rm m})_1/2$, as shown in Section 2, which is about the level required for efficient acceleration of mildly relativistic particles.

However, in an oblique shock with $|\sin\phi_1|\lesssim(8/r)(\pi\beta)^{-1/4}$, there must be a range of velocity within which an accelerating particle must be scattered by non-propagating reactive modes (region 4 of Fig. 4) before it can be scattered by the Alfvén modes in region 1. In this case:

$$\frac{\Gamma_{\rm dif}}{\Gamma_{\rm sat}}\lesssim\frac{3\pi}{4}\gamma I_{\rm m}\left(\frac{\varkappa}{\sqrt{\pi}\theta_2^2}\right)\left(\frac{\sqrt{\pi}\theta_2^2}{\sqrt{\beta}}\right)^2. \tag{4.8}$$

In the region $\varkappa\lesssim\sqrt{\pi}\theta_2^2$ a level of turbulence $I_{\rm m}$ of order unity may be required when $|\sin\phi_1|\simeq[8/r(\pi\beta)^{1/4}]$ in order for particles to be accelerated efficiently.

There is a common conclusion that can be drawn from these estimates. This is that the level of wave turbulence behind the shock must satisfy $I_{\rm m}\gtrsim\gamma^{-2}$ if suprathermal particles are to be accelerated up to a momentum $p_{\rm min}=M_{\rm i}v_{\rm i_2}\beta_2^{1/2}$. (Note that if $v_{\rm i_2}\lesssim\beta_2^{-1/2}c$ then we require that the wave turbulence be strongly non-linear.) Acceleration to higher energies can proceed with a lower level of wave turbulence as the resonant waves have longer wavelengths and so are less efficiently damped.

This conclusion should be regarded with considerable caution for several reasons. First, our results depend upon a simple model of the influence of particle trapping which may be inadequate. Secondly, we have ignored the possibility of ongoing production of long-wavelength turbulence by the shock front itself. In particular, the microscopic processes responsible for dissipating the bulk kinetic energy in the collisionless shock might lead to a temperature anisotropy in the downstream thermal plasma, leading to a (linear) instability of the waves once they cross the shock. For quasi-parallel shocks it is most likely that $T_\parallel>T_\perp$ since it is easier to increase the energy of particle motion along the ambient magnetic field. As we discuss further in Appendix A, a small pressure anisotropy $\Delta P/P\cong\min(\omega/\Omega_{\rm i},\ \varkappa^2)\ll 1$ drastically changes the dispersion relation of the waves. Unstable modes appear when $\Delta P/P\gtrsim\beta^{-1}$. If this is the case, then some distance behind the shock, scattering on thermal particles will probably lead to a (non-linearly) marginally stable state ($\Gamma \simeq 0$). Alternatively, scattering waves may be created by inhomogeneity in the unshocked medium via a turbulent cascade.

Finally, there is the possibility that the saturated wave intensities that are established ahead of the shock and transmitted across the shock are so large that a quasi-linear treatment is invalid. Sufficiently strong turbulence may not be damped efficiently, particularly in a high β plasma where the magnetic stresses are such a small fraction of the Reynolds' stresses that drive the flow.

If the damping is effective behind the shock, then the cosmic ray structure will presumably be time-dependent. In this case, we expect that cosmic rays will establish density gradients in the direction of the shock from either side and thereby generate scattering Alfvén waves. As the scattering behind the shock becomes more efficient so the cosmic ray gradient there will diminish, the waves damp and eventually the cosmic rays will be released downstream. This limit cycle behaviour would probably recur at a rate dictated by the wave growth and damping times at mildly relativistic proton energy.

If post-shock wave damping is ineffective, then there is an additional effect which should be included in the analysis. As pointed out by Bell (1978), the fact that the Alfvén waves are propagating through the fluid implies that the scattering will be inelastic. The efficiency of the acceleration will therefore be affected and the slope of the particle spectrum given by equation (1.1) will be changed. If we know the directions and intensities of scattering waves, then we can estimate this correction. For example, consider the case of the parallel shock. The Alfvén waves ahead of the shock recede from it at the Alfvén speed, a_1. The downstream scatterers will approach the shock with an average speed $\bar{\Delta}$ relative to the fluid given by the intensity-weighted average of the speeds of the reflected and transmitted waves.

$$\bar{\Delta}=\left\{\frac{(r+r^{1/2})^2-(r-r^{1/2})^2}{(r+r^{1/2})^2+(r+r^{1/2})^2}\right\}A_2=\frac{2a_1}{1+r} \tag{4.9}$$

(*cf.* Skilling 1975). The transmitted spectral slope is then corrected to

$$q=\frac{3(V_1-a_1)}{(V_1-a_1)-(V_2-\bar{\Delta})}=\left(\frac{3r}{r-1}\right)\left(1-\frac{1}{(r+1)M_{\mathrm{A}}}\right)+\mathrm{O}(M_{\mathrm{A}}^{-2}). \tag{4.10}$$

Note that the acceleration efficiency is enhanced and the slope *flattened*. If, alternatively, we assume that the downstream scatterers are at rest with respect to the fluid, the spectrum is *steepened*

$$q=\left(\frac{3r}{r-1}\right)\left(1+\frac{1}{(r-1)M_{\mathrm{A}}}\right)+\mathrm{O}(M_{\mathrm{A}}^{-2}). \tag{4.11}$$

This last case is of particular interest for low-energy particles interacting with the non-propagating modes ($\omega \simeq \mathbf{k}\cdot\mathbf{U}$) behind an oblique shock.

In this paper we have attempted to analyse the fate of scattering Alfvén waves after they cross a shock front. Our conclusion is that under most circumstances, it is necessary that the waves achieve non-linear amplitude ahead of the shock or be freshly generated behind the shock if they are to backscatter suprathermal particles and thereby accelerate new cosmic rays. This may invalidate our quasi-linear analysis. Nevertheless it is hoped that the considerations presented in this paper may already be of relevance to the interpretation of interplanetary shock wave data and may allow the inclusion of hydromagnetic wave turbulence in improved models of high Mach number collisionless shocks.

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References

Achterberg, A., 1981a. *Astrophys. J.,* **97,** 259.
Achterberg, A., 1981b. *Astr. Astrophys.,* **98,** 161.
Akhiezer, A. I., Akhiezer, I. A., Polovin, R. V., Sitenko, A. G. & Stepanov, K. N., 1975. *Plasma Electrodynamics*, Vol. II, Pergamon Press, Oxford.
Anderson, Y. E., 1963. *Magnetohydrodynamic Shock Waves*, MIT Press, Cambridge, Massachusetts.
Axford, W. I., 1981. *Plasma Astrophysics*, p. 425, eds Guyenne & Leary, ESA, Paris.
Bell, A. R., 1978. *Mon. Not. R. astr. Soc.,* **1872,** 147.
Blandford, R. D. & Eichler, D., 1985. *Phys. Repts,* submitted.
Drury, L. O'C, 1983. *Repts Prog. Phys.,* **46,** 973.
Foote, E. A. & Kulsrud, R. M., 1979. *Astrophys. J.,* **233,** 302.
Krall, N. A. & Trivelpiece, A. W., 1983. *Principles of Plasma Physics*, McGraw-Hill, New York.
Kennel, C. F., Edmiston, J. P. & Hada, T., 1984. *Proceedings of Chapman Conference on Collisionless Shocks in the Heliosphere*, Napa Valley, California, in press.
Landau, L. D. & Lifshitz, E. M., 1960. *Electrodynamics of Continuous Media,* Addison-Wesley, Reading, Massachusetts.
Lee, M. A. & Völk, H. J., 1973. *Astrophys. Space Sci.,* **24,** 31.
McKenzie, J. E. & Westphal, K. O., 1969. *Planet. Space Sci.,* **17,** 1029.
Melrose, D. B., 1980. *Plasma Astrophysics,* Vol. I, Gordon and Breach, London.
Miyamoto, K., 1980. *Plasma Physics for Nuclear Fusion*, MIT Press, Cambridge, Massachusetts.
Sivukhin, P. V., 1963. In: *Reviews of Plasma Physics*, p. 1, Vol. I, ed. Leontovich, M. A., Consultants Bureau, New York.
Skilling, J. A., 1975. *Mon. Not. R. astr. Soc.,* **172,** 557.
Stepanov, K. N., 1958. *Soviet Phys. JETP,* **34,** 892.
Völk, H. J. & Cesarsky, C. J., 1982. *Z. Naturfasch,* **37a,** 809.

Appendix A

In a high β plasma, short-wavelength waves differ substantially from the well-known cold-plasma MHD modes. Ths is due to the finite Larmor radii of the thermal ions. The relevant dispersion relation was first derived by Foote & Kulsrud (1979). Thermal electrons have a high mobility along the ambient magnetic field and can short out any parallel wave electric field (i.e. $E_{\parallel}=0$). The wave properties are then determined by the 'perpendicular' components of the dielectric tensor $\tilde{\varepsilon}$ with respect to the magnetic field. Taking the ambient magnetic field along the z-axis, and the wave vector $\mathbf{k}$ in the x–z plane, the dispersion relation can be written as ($\mathbf{k}=k_{\parallel}\hat{\mathbf{z}}+k_{\perp}\hat{\mathbf{x}}$):

$$\left(\frac{c^2k_{\parallel}^2}{\omega^2}-\varepsilon_{xx}\right)\left(\frac{c^2k^2}{\omega^2}-\varepsilon_{yy}\right)=|\varepsilon_{xy}|^2. \tag{A1}$$

(Note the change of notation from Section 2.) Generalizing the results of Foote & Kulsrud (1979) to allow for temperature anisotropies, the components of $\tilde{\varepsilon}$ are given by (to lowest order in $1/\beta$):

$$\varepsilon_{xx}=1+\frac{c^2}{a^2}\left\{1+x^2+\frac{3}{2}\varkappa^2\left(2-\frac{T_\perp}{T_\parallel}-\frac{1}{4}\frac{T_\perp}{T_\parallel}\tan^2\theta\right)+\frac{1}{2}\frac{\varkappa^2}{x^2}\left(1-\frac{T_\perp}{T_\parallel}\right)\right\}$$

$$\varepsilon_{yy}=\varepsilon_{xx}-2\frac{c^2}{a^2}\left\{\left(\frac{T_\perp}{T_\parallel}\right)^2\tan^2\theta-i\frac{\sqrt{\pi}}{2}\left(\frac{T_\perp}{T_\parallel}\right)^2\frac{|\varkappa|\tan^2\theta}{x}+\frac{\varkappa^2\tan^2\theta}{x^2}\frac{T_\perp}{T_\parallel}\left(1-\frac{T_\perp}{T_\parallel}\right)\right\}$$

$$\varepsilon_{xy}=i\frac{c^2}{a^2}\left[x+\frac{\varkappa^2}{2x}\left\{3-2\frac{T_\perp}{T_\parallel}\left(1+\frac{3}{4}\tan^2\theta y\right)\right\}\right]. \tag{A2}$$

Here we have defined $x=\omega/\Omega_\mathrm{i}$ with Ω_i the ion Larmor frequency, and $\varkappa=k_\parallel\sqrt{2T_\parallel/m_\mathrm{i}}/\Omega_\mathrm{i}$ and assumed equality of electron and ion temperatures.

For low-frequency ($x\ll1$) long-wavelength waves ($\varkappa\ll1$) propagating nearly along the magnetic field ($\theta\ll1$), the dispersion relation (A1) can be written in the form:

$$D_\mathrm{A}D_\mathrm{M}=C^2. \tag{A3}$$

where

$$D_\mathrm{A}=\nu^2-\left\{1-\beta\left(1-\frac{T_\perp}{T_\parallel}\right)\right\}$$

$$D_\mathrm{M}=D_\mathrm{A}+i\sqrt{\pi\beta}\left(\frac{T_\perp}{T_\parallel}\right)^2\frac{l}{|l|}\nu\theta^2$$

$$C=\frac{l\nu}{2}\left(3-2\frac{T_\perp}{T_\parallel}\right) \tag{A4}$$

and where we introduced $\nu=\omega/k_\parallel a$, $l=\sqrt{\beta}\varkappa$, and $\beta=8\pi nT_\parallel/B^2$. In a *low* β plasma the coupling term C can be neglected. In this case (A3) yields the well-known (undamped) Alfvén waves ($D_\mathrm{A}=0$) and the transit-time damped magnetosonic mode ($D_\mathrm{M}=0$).

Before solving equation (A3) we give an alternative derivation using guiding centre theory. This gives some insight into the character of the modes. For low-frequency waves, particle orbits can be approximated by that of their guiding centre. Neglecting displacement currents, the relevant equation governing the waves is

$$\nabla\times\delta\mathbf{B}=\frac{4\pi}{c}(\mathbf{j}_\mathrm{D}+\mathbf{j}_\mathrm{M}). \tag{A5}$$

Here $\delta\mathbf{B}$ is the wave magnetic field and $\mathbf{j}_\mathrm{D}$ and $\mathbf{j}_\mathrm{M}$ are the drift and magnetization currents respectively. Defining a guiding-centre distribution so that $n_\sigma F_\sigma(\mu,\, v_\parallel)\,d\mu\,dv_\parallel$ denotes the number of particles of species σ with velocity $v_\parallel$ along the magnetic field in the range $v_\parallel$, $v_\parallel+dv_\parallel$ and magnetic moment $\mu=P_\perp^2/2mB$ in the range μ, $\mu+d\mu$, these currents are given by:

$$\mathbf{j}_\mathrm{D}=\sum_\sigma\int dv_\parallel d\mu n_\sigma q_\sigma F_\sigma(\mu,\, v_\parallel)\,\mathbf{V}_\mathrm{D}^\sigma$$

$$\mathbf{j}_\mathrm{M}=-c\sum_\sigma\int dv_\parallel d\mu\,\nabla\times\frac{n_\sigma\mu F_\sigma(\mu,\, v_\parallel)}{B}\mathbf{B}, \tag{A6}$$

where q_σ is the particle charge, and $\sigma=e, i$ for electrons (ions), $\mathbf{V}_\mathrm{D}$ is the guiding centre velocity.

We will only need the components perpendicular to the ambient magnetic field. After some manipulation one can write the mean drift velocity $V_{\rm D}^{g}$ in the form:

$$(\mathbf{V}_{\rm D}^{g})_{\perp}=\frac{c}{B}\left\{\mathbf{E}-\left(\frac{m_\sigma c}{q_\sigma B}\right)^2\ddot{\mathbf{E}}\right\}\times\hat{\mathbf{b}}+\frac{c}{B}\left(\frac{m_\sigma c}{q_\sigma B}\right)\dot{\mathbf{E}}_{\perp}+\left(\frac{\mu c}{q_\sigma B}+\frac{m_\sigma v_\parallel^2 c}{q_\sigma B^2}\right)\hat{\mathbf{b}}\times\nabla B+\frac{mcv_\parallel^2}{q_\sigma B^2}\nabla\times\mathbf{B}_{\perp}, \tag{A7}$$

cf. Miyamoto (1980). Here $\hat{\mathbf{b}}$ is the unit vector along the magnetic field, $\dot{\mathbf{E}}\equiv\partial_t\mathbf{E}+v_\parallel(\hat{\mathbf{b}}\cdot\nabla)\mathbf{E}$ the total derivative along the (unperturbed) guiding centre orbit, and $\mathbf{A}_\perp\equiv\mathbf{A}-(\hat{\mathbf{b}}\cdot\mathbf{A})\hat{\mathbf{b}}$ for a vector $\mathbf{A}$.

Using (A6) and (A7) in (A5) yields:

$$\left\{1+\sum_\sigma\frac{4\pi(P_{\perp\sigma}-P_{\parallel\sigma})}{B^2}\right\}(\nabla\times\mathbf{B})_\perp$$

$$=\sum_\sigma\frac{4\pi n_\sigma q_\sigma}{B}\int dv_\parallel d\mu\left[\left\{\mathbf{E}-\left(\frac{m_\sigma c}{q_\sigma B}\right)^2\ddot{\mathbf{E}}\right\}\times\hat{\mathbf{b}}+\frac{m_\sigma c}{q_\sigma B}\dot{\mathbf{E}}_\perp+\frac{m_\sigma v_\parallel^2}{q_\sigma B}\hat{\mathbf{b}}\times\nabla B+\mu\hat{\mathbf{b}}\times\nabla\right]F_\sigma. \tag{A8}$$

Here we have defined

$$\left.\begin{matrix}P_{\perp\sigma}\\ P_{\parallel\sigma}\end{matrix}\right\}=\int dv_\parallel d\mu n_\sigma\left\{\begin{matrix}\mu B\\ m_\sigma v_\parallel^2\end{matrix}\right\}F_\sigma(\mu,v_\parallel), \tag{A9}$$

the perpendicular and parallel pressures of particle species σ. As long as the plasma is quasi-neutral (i.e., $\Sigma_\sigma n_\sigma q_\sigma=0$) the first term on the right-hand side of (A8) vanishes ($\mathbf{E}\times\hat{\mathbf{b}}$ drift term). Now assuming a plane wave perturbation $\mathbf{B}=\mathbf{B}_0\hat{z}+\delta\mathbf{B}\exp(i\mathbf{k}\cdot\mathbf{x}-i\omega t)$ with a wave vector $\mathbf{k}=k_\perp\hat{\mathbf{x}}+k_\parallel\hat{\mathbf{z}}$ satisfying $|k_\perp/k_\parallel|\equiv\tan\theta\ll1$ we can expand (A8) to first order in $\delta\mathbf{B}$. The perturbation in f_σ follows from the Vlasov equation for f_σ

$$\frac{\partial F_\sigma}{\partial t}+v_\parallel\hat{\mathbf{b}}\cdot\nabla F_\sigma-\frac{\mu}{m}\hat{\mathbf{b}}\cdot\nabla B\frac{\partial F_\sigma}{\partial v_\parallel}=0 \tag{A10}$$

which, upon linearization, yields

$$\delta F_\sigma=-\frac{\mu}{m}\frac{k_\parallel\,\partial F_{0\sigma}/\partial v_\parallel}{\omega-k_\parallel v_\parallel}\left(\frac{-k_\perp\delta B_x}{k_\parallel}\right)$$

where we used that $B-B_0\approx\delta B_z=-(k_\perp/k_\parallel)\delta B_x$, the last relation following from $\nabla\cdot\mathbf{B}=0$. Neglecting terms of order $|\theta|$ with respect to unity, and eliminating the wave electric field $\delta\mathbf{E}$ in terms of the wave magnetic field, the resulting equation can be written

$$D_{\rm M}\delta B_x+i\frac{\nu l}{2}\delta B_y=0$$

$$D_{\rm A}\delta B_y-i\frac{\nu l}{2}\delta B_x=0. \tag{A12}$$

We have assumed that the unperturbed guiding centre distribution is an anisotropic Maxwellian, i.e.

$$F_{0\sigma}(\mu,v_\parallel)=\frac{B}{T_\perp}\left(\frac{m}{2\pi T_\parallel}\right)^{1/2}\exp\{-(\mu B/T_\perp+m_\sigma v_\parallel^2/2T_\parallel)\}.$$

The coupling term between the two polarizations (δB_x: magnetosonic mode, δB_y, Alfvén mode) comes from the $\ddot{\mathbf{E}}$ term in (A8), specifically one has $\delta\ddot{\mathbf{E}}\times\hat{\mathbf{b}}\approx+\omega/k_\parallel c(\omega-k_\parallel v_\parallel)^2\delta\mathbf{B}_\perp\approx(\omega/k_\parallel c)k_\parallel^2 v_\parallel^2\delta\mathbf{B}_\perp$ when $\omega\ll k_\parallel v_\parallel$ as is the case here (*cf.* Foote & Kulsrud 1979). The imaginary part in D_{M} [see (A4)] results from the reactive response in the magnetization current (last term in the right-hand side of A8) due to the perturbation in the density $\delta n_\sigma=\int d\mu\,dv_\parallel\delta f_\sigma$, with δf_σ given by (A11), using the Plemelj rule $(\omega-k_\parallel v_\parallel+i\mathrm{O}^+)^{-1}=P\{(\omega-k_\parallel v_\parallel)^{-1}\}+\pi i\delta(\omega-k_\parallel v_\parallel)$ (e.g., Krall & Trivelpiece 1983, p. 378). The resulting resonant bunching of particles along the ambient field under influence of the $-\mu\nabla B$ force induced by the wave causes the transit-time damping.

This (simplified) derivation reproduces the result (A3) for the dispersion relation which was derived using a full kinetic treatment, as long as $T_\perp/T_\parallel=\mathrm{O}(1)$. (But note that only a very *small* anisotropy $T_\perp/T_\parallel-1=\mathrm{O}(1/\beta)\ll1$ is needed to drive a firehose instability; see below.)

We now proceed to solve dispersion relation (A3). Defining a quantity $\tilde{D}=D_{\mathrm{A}}+i(\sqrt{\pi\beta}/2)(T_\perp/T_\parallel)^2(l/|l|)\nu\theta^2$, (A3) can be written in the form $\tilde{D}^2+\tfrac14(T_\perp/T_\parallel)^4\pi\beta\theta^4\nu^2=C^2\equiv(l^2\nu^2/4)\zeta^2$, where we have defined $\zeta=3-2(T_\perp/T_\parallel)$. This equation can be formally solved by $\tilde{D}=\pm(l\nu/2)\sqrt{\zeta^2-\sigma^2}$, where $\sigma\equiv\sqrt{\pi\beta}\theta^2(T_\perp/T_\parallel)^2/|l|$. Using the definition for $\tilde{D}$ and D_{A} (equation A4) we obtain the following quadratic equation for $\nu=\omega/k_\parallel a$:

$$\nu^2+(i\sigma\pm\sqrt{\zeta^2-\sigma^2})\frac{l\nu}{2}-\left\{1-\beta\left(1-\frac{T_\perp}{T_\parallel}\right)\right\}=0. \tag{A13}$$

This shows that there are four independent modes, and a formal solution of (A13) yields:

$$\nu=-i\,\frac{\sigma l}{4}\{1+is\sqrt{(\zeta^2/\sigma^2)-1}\}\pm\left[\left\{1-\beta\left(1-\frac{T_\perp}{T_\parallel}\right)\right\}-\frac{l^2\sigma^2}{4}\left\{1+is\sqrt{(\zeta^2/\sigma^2)-1}\right\}^2\right]^{1/2}. \tag{A14}$$

Here $s=\pm1$. In these equations σ is a measure for the importance of transit-time damping whereas ζ is a measure for the coupling between the two (Alfvén-like and magnetosonic) polarizations in a high β plasma.

First we look at the '*weak coupling*' case with $\zeta^2/\sigma^2\ll1$, or equivalently $\sqrt{\pi\beta}\tan^2\theta\gg|l|\zeta(T_\parallel/T_\perp)^2$. We then expand $1+is\sqrt{(\zeta^2/\sigma^2)-1}\approx1-s+s(\zeta^2/2\sigma^2)$. Equation (A14) then yields

$$s=+1:\ \nu=\frac{\omega}{k_\parallel a}\simeq-\frac{il\zeta^2}{8a}\pm\left\{1-\beta\left(1-\frac{T_\perp}{T_\parallel}\right)-\left(\frac{l\zeta^2}{8\sigma}\right)^2\right\}^{1/2} \tag{A15a}$$

$$s=-1:\ \nu=\frac{\omega}{k_\parallel a}\simeq-i\frac{l\sigma}{2}\pm\left\{1-\beta\left(1-\frac{T_\perp}{T_\parallel}\right)-\frac{l^2\sigma^2}{4}\right\}^{1/2}. \tag{A15b}$$

The $s=+1$ case corresponds to the weakly damped Alfvén wave. It becomes *reactive* {i.e., $\mathrm{Re}(\nu)=0$, $\mathrm{Im}(\nu)\leq0$} if $l\zeta^2/8\sigma>\sqrt{1-\beta\{1-(T_\perp/T_\parallel)\}}$, and *firehose unstable* (mode with + sign) if $1-\beta\{1-(T_\perp/T_\parallel)\}<0$. The $s=-1$ case is the magnetosonic wave, damped by regular transit-time damping. It becomes reactive when $l\sigma/2>\sqrt{1-\beta\{1-(T_\perp/T_\parallel)\}}$, and has a firehose–unstable mode if $1-\beta\{1-(T_\perp/T_\parallel)\}<0$.

These reactive modes are given by:

Reactive Alfvén-like:

$$(\nu_{\mathrm{A}}^R)_1=-il\zeta^2/4\sigma \tag{A16a}$$

$$(\nu_{\mathrm{A}}^R)_2=-i\,\frac{4\sigma}{l\zeta^2}\left\{1-\beta\left(1-\frac{T_\perp}{T_\parallel}\right)\right\} \tag{A16b}$$

Reactive magnetosonic:

$$(\nu_{\rm M}^R)_1 = -il\sigma \tag{A16c}$$

$$(\nu_{\rm M}^R) = -\frac{i}{l\sigma}\left\{1-\beta\left(1-\frac{T_\perp}{T_\parallel}\right)\right\}. \tag{A16d}$$

In the *strong coupling* case, where $\zeta^2/\sigma^2 \gg 1$, one can expand (A14) in σ/ζ. To lowest order:

$$\nu = s\frac{l\zeta}{4}\left(1-is\frac{\sigma}{\zeta}\right) \pm \left\{1-\beta\left(1-\frac{T_\perp}{T_\parallel}\right)+\frac{l^2\zeta^2}{16}\left(1-is\frac{\sigma}{\zeta}\right)^2\right\}^{1/2}. \tag{A17}$$

In the *long-wavelength limit* ($l\zeta/4 \ll 1$) one finds a 'fast' (i.e. $|\nu| \geq 1$) and a 'slow' (i.e. $|\nu| \leq 1$) mode, depending on the signs; choosing $\mathrm{Re}(\nu)$ positive one finds:

$$\nu = \left\{1-\beta\left(1-\frac{T_\perp}{T_\parallel}\right)+\frac{l^2\zeta^2}{16}\right\}^{1/2} \pm \frac{l\zeta}{4} - i\frac{l\sigma}{4}. \tag{A18}$$

They can be thought of as modified Alfvén and magnetosonic waves.

In the *short wavelength limit* $[l\zeta/4 \gg 1-\beta\{1-(T_\perp/T_\parallel)\}]$ one finds:

$$\nu = \pm\frac{l\zeta}{2} - i\frac{l\sigma}{2} \tag{A19a}$$

$$\nu = \pm\frac{2[1-\beta\{1-(T_\perp/T_\parallel)\}]}{l\zeta}\left(1 \mp i\frac{\sigma}{\zeta}\right) \tag{A19b}$$

again corresponding to a fast and slow mode respectively. It can easily be seen from these equations that firehose-unstable modes exist when $1-\beta\{1-(T_\perp/T_\parallel)\}<0$.

For the weakest damped modes in absence of a firehose instability one sees that the following interpolation formula is valid for the damping rate:

$$-\mathrm{Im}(\nu) = \frac{\Gamma_{\rm L}\Omega_{\rm i}}{2|k_\parallel|a} = \frac{|l|\sigma/2}{(1+l^2\zeta^2/4)(1+4\sigma^2/\zeta^2)}. \tag{A20}$$

This corresponds to equation (3.1) of the main paper. In Figs 4, 5 the various results derived in this Appendix are represented graphically.

Appendix B

In this Appendix, we calculate the linear transit-time damping for obliquely propagating magnetosonic waves. Since the waves under consideration were originally generated in the upstream region by high-energy particles, their wavelength is long compared with the gyroradius of the thermal ions, $r_{\rm L}$. $p=\gamma mv = m\Omega_{\rm i}/k_\parallel$ is the typical momentum of the resonant, suprathermal particles. The motion of a particle in the wave electromagnetic field can be treated in the guiding centre approximation. Denoting by $p_\parallel$ and $p_\perp$ the magnitude of the components of momentum of a particle along and perpendicular to the ambient magnetic field, the relevant equations of motion read (Sivukhin 1965):

$$\frac{dp_\perp}{dt} = \frac{p_\perp}{2B}\left(\frac{\partial B}{\partial t}+v_\parallel\frac{\partial B}{\partial s}\right)$$

$$\frac{dp_\parallel}{dt} = -\frac{p_\perp}{2B}v_\perp\frac{\partial B}{\partial s}+qE_\parallel. \tag{B1}$$

Here B is the *magnitude* of the magnetic field, and s is a coordinate along the field; $E_{\|}$ is the component of the electric field along the magnetic field. We will work in the rest-frame of the plasma.

Slow magnetosonic waves at long wavelengths ($kr_L < 1/\sqrt{\beta}$) and all modes at short wavelengths ($kr_L \gtrsim (1/\sqrt{\beta})$ have a component of the wave magnetic field along the ambient field (see Table 1). This component induces a linear perturbation in the magnetic field strength $B = |\mathbf{B}_0 + \delta\mathbf{B}|$:

$$\delta B \cong \frac{\mathbf{B}_0 \cdot \delta\mathbf{B}_S}{B_0} = \frac{k_\perp}{k} \delta B_S + O\left(\frac{\delta B_S^2}{B_0}\right). \tag{B2}$$

Here $k_\perp$ and $k_{\|}$ are the components of the wave vector perpendicular to and along the ambient magnetic field $\mathbf{B}_0$, $k = \sqrt{k_\perp^2 + k_{\|}^2}$.

Assuming a superposition of plane waves:

$$\delta\mathbf{B}_S = \int \frac{d\mathbf{k}}{(2\pi)^3} \mathbf{b}_S(\mathbf{k}) \exp[i\mathbf{k}\cdot\mathbf{x} - i\omega t],$$

the interaction between a particle and the slow mode waves takes place at the Cherenkov resonance

$$\omega - k_{\|} v_{\|} = 0, \tag{B3}$$

where the phase of wave as seen by the particle (or more precisely its guiding centre) is stationary. Only then is there an irreversible exchange of energy between the wave and the particle. This immediately implies that $dp_\perp/dt \propto \omega - k_{\|} v_{\|} = 0$. The result of the interaction is the well-known quasi-linear diffusion in parallel momentum (e.g. Melrose 1980; see also Achterberg 1981a), which changes the distribution function $f(p_\perp, p_{\|})$ according to:

$$\frac{\partial f(p_\perp, p_{\|})}{\partial t} = \frac{\partial}{\partial p_{\|}} \left\{ D_{\|\|} \frac{\partial f(p_\perp, p_{\|})}{\partial p_{\|}} \right\} \tag{B4}$$

where

$$D_{\|\|} \equiv \frac{\pi}{4} \frac{p_\perp^2 v_\perp^2}{B_0^2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{k_{\|}^2 k_\perp^2}{k^2} |b_S(\mathbf{k})|^2 \delta(\omega_k - k_{\|} V_{\|}). \tag{B5}$$

Here we have used the fact that $\delta E_{\|} = 0$ for these waves. The electron contribution to the damping is reduced by a factor $\sqrt{(m_e/m_i)(T_e/T_i)}$, and can generally be neglected.

The damping rate is now obtained by a simple argument of energy conservation:

$$\int d^3p \, ½ m V^2 \frac{\partial f}{\partial t} + \frac{\partial}{\partial t} \int \frac{d^3\mathbf{k}}{(2\pi)^3} W_S(k) = 0, \tag{B6}$$

with $\partial f/\partial t$ determined by (25) and (26), and the wave energy density $W_S(k)$ of the waves in $\mathbf{k}$-space as usual defined by (for a high β plasma):

$$W_S(k) = ½ n_0 m_i |V_p^2| + \frac{1}{8\pi} |b_s|^2 \simeq \left(\frac{\omega^2}{k_{\|}^2 a^2} + 1\right) \frac{|b_s(\bar{k})|^2}{8\pi}. \tag{B7}$$

Here we used the fact that most of the kinetic energy (to order kR_L, $\omega/\Omega_i \ll 1$) is in the $\mathbf{E}\times\mathbf{B}$ drift $v_D \simeq (c\bar{E} \times \bar{B}_0/B_0^2) \simeq (\omega/k\|)(\delta B_\perp/B_0)$ of the ions. Assuming a Maxwellian distribution for the thermal plasma, a simple calculation yields a linear damping rate $\Gamma_L \equiv -\partial \ln W^S(k)/\partial t$ given by:

$$\Gamma_L = \pi^{1/2} |k_\parallel| v_i \sin^2\theta \frac{2\omega^2}{\omega^2 + k_\parallel^2 a^2}. \tag{B8}$$

Here θ is defined by $\cos\theta = (\mathbf{k}\cdot\mathbf{B})/|\mathbf{k}||\mathbf{B}|$. This reproduces the expressions derived from the dispersion relation in Appendix A.

Appendix C

We consider non-linear transit-time damping. Representing the wave field as a superposition of plane waves, i.e.

$$\delta\mathbf{B} = \int \frac{d\mathbf{k}\, d\omega}{(2\pi)^4} \mathbf{b}(\mathbf{k}, \omega) \exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t), \tag{C1}$$

with $\mathbf{b}\cdot\mathbf{B}_0 = 0$ ($\mathbf{B}_0$ is the ambient magnetic field), these waves induce a variation in the magnetic field strength $B = |\mathbf{B}_0 + \delta\mathbf{B}|$ given by:

$$\delta B = \int \frac{d\mathbf{k}\, d\omega}{(2\pi)^4} \int \frac{d\mathbf{k}'\, d\omega}{(2\pi)^4} \frac{\{\mathbf{b}(\mathbf{k}, \omega)\cdot\mathbf{b}(\mathbf{k}'\cdot\omega')\}}{2B_0} \exp\{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x} - i(\omega+\omega')t\} \tag{C2}$$

where we assumed $|\delta\mathbf{B}|/B_0 \ll 1$.

If the wavelength and frequency of the beat waves contained in (C2) are much larger than the typical gyroradius $\varrho = v_\perp/\Omega$ and smaller than the gyrofrequency Ω respectively, we can use the gyrocentre approximation (B1) to describe the interaction between the waves and the particles. The phase of the beat waves as seen by the particle equals

$$\phi = \phi_0 + (k_\parallel + k_\parallel') v_\parallel t + (k_\perp + k_\perp')\varrho \sin(\Omega t) - (\omega + \omega')t, \tag{C3}$$

where we assumed that the $\mathbf{k}$ vectors lie in the same plane, containing $\mathbf{B}_0$. This allows us to expand

$$\exp(i\phi) = \sum_{S=-\infty}^{S=+\infty} J_S\left\{\frac{(k_\perp + k_\perp')v_\perp}{\Omega}\right\} \exp(i\phi_S) \tag{C4}$$

where $\phi_S = \phi_0 + \{(k_\parallel + k_\parallel')v_\parallel - (\omega+\omega') + s\Omega\}t$ and J_s the Bessel function of order s. The assumption of long wavelengths and low frequency limits resonant interactions to the $s=0$ resonance.

$$(k_\parallel + k_\parallel')v_\parallel - (\omega + \omega') = 0 \tag{C5}$$

and we can approximate

$$J_0\left\{\frac{(k_\perp + k_\perp')v_\perp}{\Omega}\right\} \simeq 1.$$

Defining a guiding centre distribution function $f(p_\perp, p_\parallel)$ so that $f(p_\perp, p_\parallel)\, dp_\parallel 2\pi p_\perp dp_\perp$ is the number of particles with parallel momentum between $p_\parallel$ and $p_\parallel + dp_\parallel$, perpendicular momentum between $p_\perp$ and $p_\perp + dp_\perp$, the Liouville equation for $f(p_\perp, p_\parallel)$ in phase space reads (*cf.* equation B4):

$$\frac{\partial f}{\partial t} + v_\parallel \frac{\partial f}{\partial z} + \frac{1}{p_\perp}\frac{\partial}{\partial p_\perp}\left\{\frac{p_\perp^2}{2B}\left(\frac{\partial B}{\partial t} + v_\parallel \frac{\partial B}{\partial z}\right) f\right\} + \left(qE_\parallel - \frac{p_\perp}{2B} v_\perp \frac{\partial B}{\partial z}\right)\frac{\partial f}{\partial p_\parallel} = 0. \tag{C6}$$

Here we have oriented the ambient field along the z-axis.

First of all we calculate the parallel electric field $E_\parallel$, induced by the variations in B. (The Alfvén waves in the *linear* approximation have $E_\parallel=0$). Defining a potential ϕ so that $E_\parallel=-\partial\phi/\partial z$, and expanding both f and ϕ in Fourier components, one finds after linearizing (C6) in $E_\parallel$ and δB:

$$f_{\tilde{k}}=-q\phi_{\tilde{k}}\frac{\tilde{k}_\parallel\partial f_0/\partial p_\parallel}{\tilde{\omega}-\tilde{k}_\parallel v_\parallel}-\int\frac{d\mathbf{k}\,d\omega}{(2\pi)^4}\frac{b(\mathbf{k},\omega)b(\mathbf{k}',\omega)}{4B_0^2}\left\{\frac{\tilde{k}_\parallel V\partial f_0/\partial p_\parallel}{\tilde{\omega}-\tilde{k}_\parallel V_\parallel}+\frac{1}{p_\perp}\frac{\partial}{\partial p_\perp}(p_\perp^2 f_0)\right\}. \tag{C7}$$

Here $f_0(p_\perp, p_\parallel)$ is the unperturbed distribution function, and we introduced the notation $\tilde{k}_\parallel=k_\parallel+k'_\parallel$, $\tilde{\omega}=\omega+\omega'$. Putting this into Poisson's equation:

$$\tilde{k}_\parallel^2\phi_{\tilde{k}_\parallel}-4\pi\sum_\sigma q_\sigma\int dp_\parallel dp_\perp 2\pi p_\perp f_{\tilde{k}}=0,$$

under the assumption that $f_0(p_\perp, p_\parallel)$ is a Maxwellian, leads to the following equation for $\phi_{\tilde{k}_\parallel}$

$$\left\{1+\sum_\sigma X^\sigma(\tilde{k},\tilde{\omega})\right\}\phi_{\tilde{k}}=-\sum_\sigma\frac{2T_\sigma}{q_\sigma}X^\sigma(\tilde{k},\tilde{\omega})\int\frac{d\mathbf{k}\,d\omega}{(2\pi)^4}\frac{\mathbf{b}^*(\mathbf{k},\omega)\cdot\mathbf{b}(\mathbf{k}',\omega')}{nB_0^2}. \tag{C8}$$

Here we have defined the linear (longitudinal) susceptibility for particle species σ by

$$X^\sigma(\mathbf{k},\omega)=\frac{4\pi q_\sigma^2}{k_\parallel^2}\int 2\pi p_\perp dp_\perp dp_\parallel\frac{k_\parallel\partial f_0^\sigma/\partial p_\parallel}{\omega-k_\parallel v_\parallel}. \tag{C9}$$

T_σ is the temperature of particle species σ in energy units. In a high β plasma, one has $\omega/k_\parallel V_\parallel\simeq\omega/k_\parallel\sqrt{T_\sigma/m_\sigma}\ll 1$, and the susceptibility (B8) for the electrons and ions reduces to:

$$X_k^e\simeq\frac{\omega_{ps}^2}{k_\parallel^2 v_s^2}$$

$$X_k^i\simeq\frac{\omega_{pi}^2}{k_\parallel^2 v_i^2}\left(1-\frac{\omega^2}{k_\parallel^2 v_i^2}+i\sqrt{\pi/2}\frac{\omega}{|k_\parallel|v_i}\right). \tag{C10}$$

Here $\omega_{p\sigma}^2$ is the plasma frequency of particle species σ. Using this in (C8) yields:

$$\phi^{(2)}(\tilde{\mathbf{k}},\tilde{\omega})=\frac{T_e}{2e}\frac{\tilde{\omega}}{|\tilde{\mathbf{k}}_\parallel|V_{ti}}\left(\frac{\tilde{\omega}}{|\tilde{\mathbf{k}}_\parallel|V_{ti}}-i\sqrt{\pi/2}\right)\int\frac{d\mathbf{k}\,d\omega}{(2\pi)^4}\frac{\mathbf{b}(\mathbf{k},\omega)\cdot\mathbf{b}(\mathbf{k}',\omega')}{B_0^2}. \tag{C11}$$

Here e is the proton charge, and we assumed the long-wavelength limit $k_\parallel^2V_\sigma^2/\omega_{p\sigma}^2\ll 1$.

We now can estimate the relative importance of the induced electric field $E_\parallel$ for the motion of the guiding centre along the ambient magnetic field. Comparing the two terms in equation (B1) for $dp_\parallel/dt$ using (C2) and (C11), it follows immediately that for a thermal ion, with $\langle p_\perp v_\perp\rangle\simeq 2T_i$, the force associated with the induced electric field is a factor $|\tilde{\omega}/k_\parallel V_i|(T_e/T_i)\simeq(a/V_i)(T_e/T_i)$, $(kr_L)(T_e/T_i)\ll 1$ smaller than the force due to changes in B. Therefore, in a high β plasma, the induced electric field $E_\parallel$ can be neglected when treating wave–particle interactions.

Now using standard quasi-linear theory to calculate the effect on the particle distribution one finds the analogue of equation (B4)

$$\frac{\partial f_0}{\partial t}=\frac{\partial}{\partial p_\parallel}\left\langle\frac{p_\perp v_\perp}{2B_0}\frac{\partial B}{\partial z}f\right\rangle=\frac{\partial}{\partial p_\parallel}\left(D_{\parallel\parallel}^{NL}\frac{\partial f_0}{\partial p_\parallel}\right). \tag{C12}$$

Here $\langle\ \rangle$ denotes an ensemble average, and $D_{\|\|}^{NL}$ is given by

$$D_{\|\|}^{NL}=\int\frac{d\mathbf{k}_1 d\mathbf{k}_2}{(2\pi)^6}\frac{p_\perp^2 v_\perp^2}{16B_0^4}\langle|\mathbf{b}(k_1)|^2\rangle\langle|\mathbf{b}(k_2)|^2\rangle$$
$$\times[(k_{\|1}+k_{\|2})^2\pi\delta\{\omega_1+\omega_2-(k_{\|1}+k_{\|2})V_\|\}+(k_{\|1}-k_{\|2})^2\pi\delta\{\omega_1-\omega_2-(k_{\|1}-k_{\|2})V_\|)\}]. \quad \text{(C13)}$$

In deriving (C13) we have used (C2) and (C7), used the standard assumption of weak turbulence in evaluating the four-point correlation:

$$\langle\mathbf{b}(1)\mathbf{b}(2)\mathbf{b}(3)\mathbf{b}(4)\rangle=\langle\mathbf{b}(1)\mathbf{b}(2)\rangle\langle\mathbf{b}(3)\mathbf{b}(4)\rangle+\langle\mathbf{b}(1)\mathbf{b}(3)\rangle\langle\mathbf{b}(2)\mathbf{b}(4)\rangle+\langle\mathbf{b}(1)\mathbf{b}(4)\rangle\langle\mathbf{b}(2)\mathbf{b}(3)\rangle$$

[where $\mathbf{b}(1)\equiv\mathbf{b}(\mathbf{k}_1,\omega_1)$ etc.], separated out the positive and negative frequency normal modes, e.g. for Alfvén waves

$$\langle\mathbf{b}(1)\mathbf{b}(2)\rangle=\tfrac{1}{2}\langle|\mathbf{b}(k_1)|^2\rangle(2\pi)^4\delta(\mathbf{k}_1+\mathbf{k}_2)\delta(\omega_1+\omega_2)\times\{2\pi\delta(\omega_1+|\mathbf{k}_\||_1a)+2\pi\delta(\omega_1-|\mathbf{k}_\||_1a)\}$$

and converted to *positive frequencies only* in (C13) using $\omega(-\mathbf{k})=-\omega(\mathbf{k})$ (as is required by the fact that $\delta\mathbf{B}$ is real). The two terms in (C13) then correspond to the resonant interaction between the particles and the high- and low-frequency beat waves of the two Alfvén waves with frequency $\omega_1=|k_\||_1a$, $\omega_2=|k_\||_2a$. It was also assumed that the field perturbations lie in the same plane, which is the case relevant to shock acceleration, as outlined in the main paper. The resonance condition (C5) also implies that $p_\perp$ is constant during the wave–particle interaction, *cf.* equation (B1).

The wave damping rate associated with the diffusion in momentum space (C12) can be obtained by an argument of detailed balancing. This is most easily accomplished by using the semi-classical formulation (e.g. Melrose 1980) which expresses the wave–particle interaction from a quantum-mechanical point of view, treating the waves as quasi-particles (bosons) with occupation number $N(\mathbf{k})$, momentum $\hbar\mathbf{k}$ and energy $\hbar\omega_k$. The resonance conditions in (C13) then simply express energy conservation for the process of 'induced fusion' of two waves in a particle, and scattering of wave 1 into wave 2 by a particle. In this language, $D_{\|\|}^{NL}$ can be written as (*cf.* Melrose 1980, equations 5.89, 5.98):

$$D_{\|\|}^{NL}=\int\frac{d\mathbf{k}_1}{(2\pi)^3}\frac{d\mathbf{k}_2}{(2\pi)^3}N(\mathbf{k}_1)N(\mathbf{k}_2)\{\hbar^2(k_{\|1}+k_{\|2})^2w(\mathbf{p},-\mathbf{k}_2,\mathbf{k}_1)+\hbar^2(k_{\|1}-k_{\|2})^2w(\mathbf{p},\mathbf{k}_2,\mathbf{k}_1)\}. \quad \text{(C14)}$$

Here $w(\mathbf{p},\mp\mathbf{k}_2,\mathbf{k}_1)$ is the transition probability for the two processes. We can now read off these transition probabilities from (C13) by noting that for Alfvén waves the number of wave quanta is determined by:

$$N(\mathbf{k})\hbar\omega_k\equiv W(\mathbf{k})=\frac{\langle|b(\mathbf{k})|^2\rangle}{8\pi}\left(1+\frac{\omega^2}{k_\|^2a^2}\right) \quad \text{(C15)}$$

where $W(\mathbf{k})\{d^3\mathbf{k}/(2\pi)^3\}$ is the energy density of the waves in a volume element $d^3\mathbf{k}$ around $\mathbf{k}$. This yields

$$w(\mathbf{p},\mathbf{k}_2,\mathbf{k}_1)=\frac{4\pi^2p_\perp^2V_\perp^2}{B_0^4}\frac{\omega_1\omega_2\pi\delta(\omega_1-\omega_2-[k_{\|1}-k_{\|2}]V_\|)}{(1+\omega_1^2/k_{\|2}^2a^2)(1+\omega_2^2/k_{\|2}^2a^2)}. \quad \text{(C16)}$$

The corresponding damping rate (destruction of wave quanta) using detailed balancing is given by (*cf*. Melrose 1980, equation 589, neglecting spontaneous emission):

$$\frac{dN(\mathbf{k}_1)}{dt}=\sum_\sigma \int \frac{d\mathbf{k}_2}{(2\pi)^3}\int d\mathbf{p}N(\mathbf{k}_1)N(\mathbf{k}_2)\{\hbar(k_{\|1}-k_{\|2})w(\mathbf{p},\ \mathbf{k}_2,\ \mathbf{k}_1)+\hbar(k_{\|1}+k_{\|2})w(\mathbf{p},\ -\mathbf{k}_2,\ \mathbf{k}_1)\}$$

$$\times\frac{\partial f_0^\sigma}{\partial p_\|}\equiv-\Gamma_{\mathrm{NL}}N(\mathbf{k}_1). \tag{C17}$$

Assuming, as before, a Maxwellian for f_0^σ, one can calculate the damping rate from (C14)–(C16) to be

$$\Gamma_{\mathrm{NL}}=\sqrt{\pi/2}\,|k_{\|1}|\,V_{\mathrm{ti}}\int\frac{d\mathbf{k}_2}{(2\pi)^3}\,\frac{W(\mathbf{k}_2)}{(B_0^2/8\pi)}\,g_{12}, \tag{C18}$$

where:

$$g_{12}\equiv 2\,\frac{\omega_1}{|k_{\|1}|a}\cdot\frac{(\omega_1-\omega_2)/(|k_{\|1}-k_{\|2}|a)+(\omega_1+\omega_2)/(|k_{\|1}+k_{\|2}|a)}{(1+\omega_1^2/k_{\|1}^2a^2)(1+\omega_2^2/k^2_{\|2}a^2)}.$$

For the long-wavelength ($kr_{\mathrm{L}}>1/\sqrt{\beta}$) Alfvén waves, for which non-linear transit time damping is most important since the linear damping is vanishingly small, one has:

$$g_{12}\equiv\frac{|k_{\|1}+k_{\|2}|(|k_{\|1}|-|k_{\|2}|)+|k_{\|1}-k_{\|2}|(|k_{\|1}|+|k_{\|2}|)}{2|k_{\|1}^2-k_{\|2}^2|}.$$

Note that when all **k** vectors are in the same half-space so that sign $(k_{\|1})=$ sign $(k_{\|2})$, one has $g_{12}=1$ if $|k_{\|2}|<|k_{\|1}|$, $g_{12}=0$ if $|k_{\|2}|>|k_{\|1}^2|$.